Lecture 9
THE DEFINITION OF TORSION

Plan

1. The tensional shearing stress by torsion.
2. About the angle of twist.
3. The strength - weight ratio under torsion.

9.1. The tensional shearing stress by torsion.

Consider a bar rigidly clamped at one and twisted at the other end by a torque (twisting moment):

$$T = F \cdot d$$

applied in a plane perpendicular to the axis of the bar as shown in Fig. 9.1. Such a bar is in torsion. An alternative representation of the torque is the double - headed vector directed along the axis of the bar.

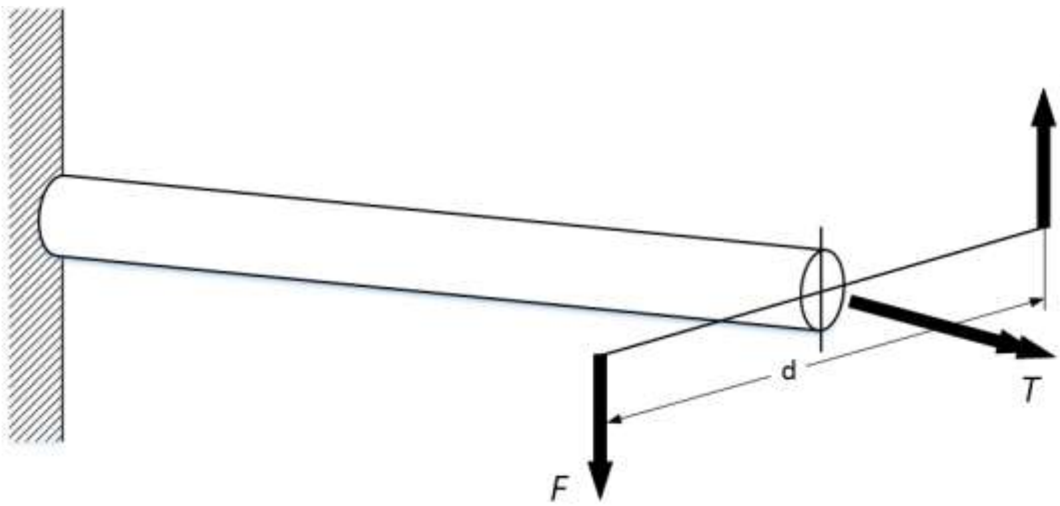

Fig. 9.1

Occasionally a number of couples act along the length of a shaft. In that case it is convenient to introduce a new quantity, the twisting moment, which fur any section along the bar is defined to be the algebraic sum of the moments of the applied couples that lie to one side of the section in question. The choice of side in any case is of course arbitrary.

For a hollow circular shaft of outer diameter d with a concentric circular hole of diameter D, the polar moment of inertia of the cross -

sectional area, usually denoted by I_ρ. is given by:

$$I_\rho = \frac{\pi}{32}\left(D^4 - d^4\right). \qquad (9.1)$$

The polar moment of inertia for a solid shaft is obtained by setting $d = 0$.

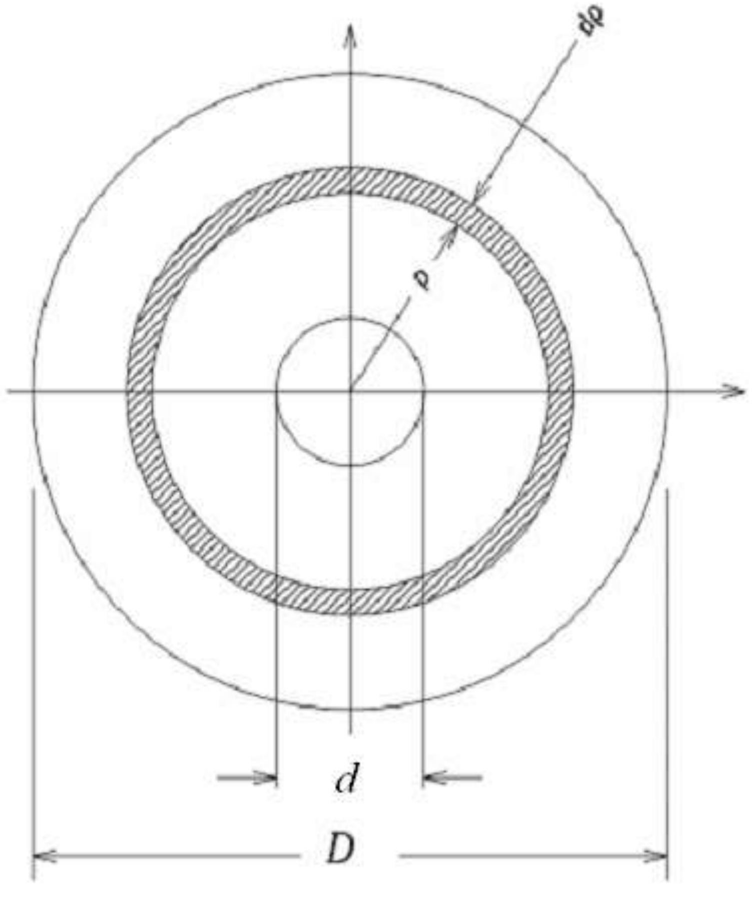

Fig. 9.2

Let D denote the outside diameter of the shaft and d, the inside diameter. Because of the circular symmetry involved, it is most convenient to adopt the polar coordinate system shown in Fig. 9.2.

By definition, the polar moment of inertia is given by the integral:

$$I_\rho = \int_A \rho^2 dA,$$

where A indicates that the integral is to be evaluated over the entire cross - sectional area.

To evaluate this integral we select as an element of area a thin ring-shaped element of radius ρ and radial thickness $d\rho$ as shown. The area of the ring is:

$$I_\rho = \int_{d/2}^{D/2} \rho^2 (2\pi\rho) d\rho = \frac{\pi}{32}\left(D^4 - d^4\right).$$

The units of I_ρ are in^4 or m$^{3.}$ For the special case of a solid circular shaft, the above becomes:

$$I_\rho = \frac{\pi d^4}{32},$$

where d denotes the diameter of the shaft.

This quantity I_ρ is a mathematical property of the geometry of the cross section which occurs in the study of the stresses set up in a circular shaft subject to torsion.

Occasionally it is convenient to rewrite the above equation in the form:

$$I_\rho = \frac{\pi}{32}\left(D^2 - d^2\right)\left(D^2 + d^2\right) = \frac{\pi}{32}(D-d)(D+d)\left(D^2 + d^2\right).$$

This last form is useful in numerical evaluation of I_ρ in those cases where the difference $(D-d)$ is small.

For either a solid or a hollow circular shaft subject to a twisting moment T the tensional shearing stress τ at a distance ρ from the center of the shaft is given by:

$$\tau = \frac{T \cdot \rho}{I_\rho}. \qquad (9.2)$$

This stress distribution varies from zero at the center of the shaft (if it is solid) to a maximum at the outer fibers, as shown in Fig. 9.3. It is to be emphasized that no points of the bar are stressed beyond the proportional limit.

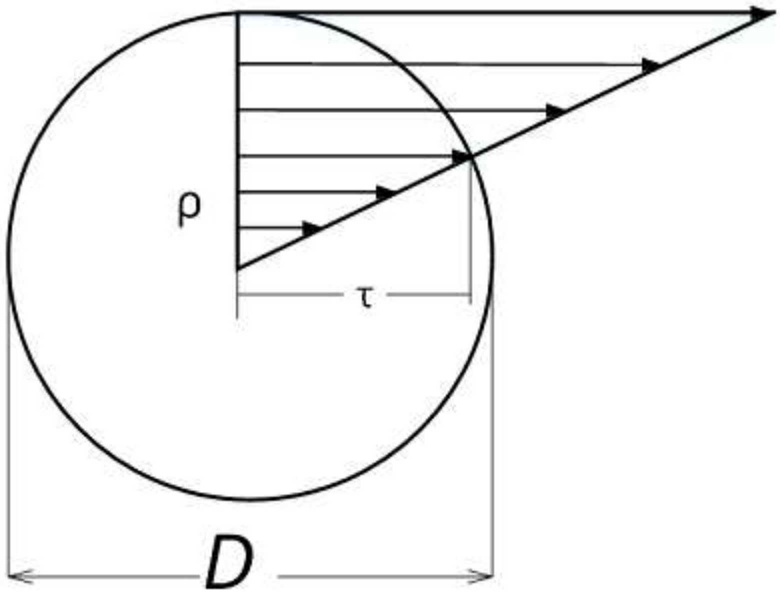

Fig.9.3

Let us derive an expression relating the applied twisting moment acting on a shaft of circular cross section and the shearing stress at any point in the shaft.

In fig. 9.4, a the shaft is shown loaded by the two torques T and consequently is in static equilibrium. To determine the distribution of shearing stress in the shaft, let us cut the shaft by a plane passing through it in a direction perpendicular to the geometric axis of the bar.

The free - body diagram of the portion of the shaft to the left of this plane appears as in fig. 9.4, b.

Obviously a torque T must act over the cross section cut by the plane. This is true since the entire shaft is in equilibrium, and hence any portion of it also is. The torque T acting on the cut section represents the effect of the right portion of the shaft on the left portion. Since the right portion has been removed, it must be replaced by its effect on the left portion. This effect is represented by the torque T. This torque is of course a resultant of shearing stresses distributed over the cross section. It is now necessary to make certain assumptions in order to determine the nature of the variation of shear stress intensity over the cross section.

One fundamental assumption is that a plane section of the shaft normal to its axis before loads are applied remains plane and normal to the axis after loading. This may be verified experimentally for circular shafts, but this assumption is not valid for shafts of noncircular cross section.

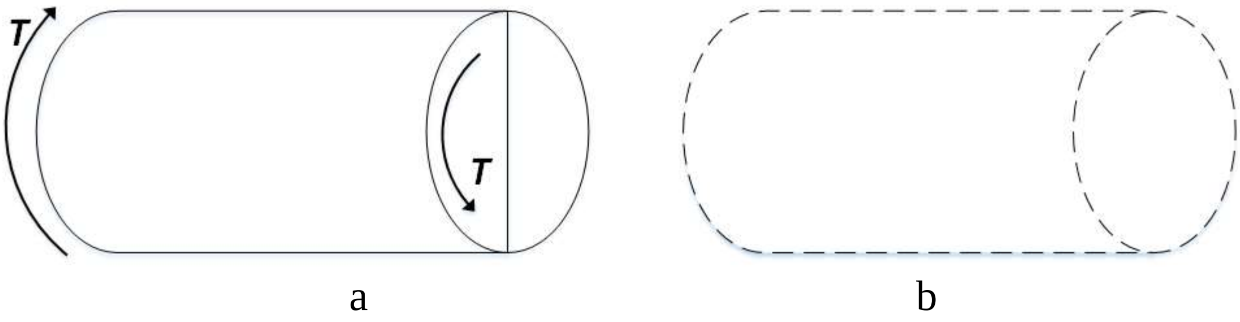

Fig. 9.4

A generator on the surfacc of the shaft, denoted by O_1A in Fig. 9.5. deforms into the configuration O_1B after torsion has occurred. The angle between these configurations is denoted by α. By definition, the shearing unit strain γ on the surface of thc shaft is:

$$\gamma = tg\alpha \approx \alpha,$$

where the angle α is measured in radians. From the geometry of the figure

$$\alpha = \frac{AB}{\ell} = \frac{r\theta}{\ell}.$$

Hence

$$\gamma = \frac{r\theta}{\ell}.$$

But since a diameter of the shaft prior to loading is assumed to remain a diameter after torsion has occurred, the shearing unit strain at a general distance ρ from the center of the shaft may likewise be written:

$$\gamma = \frac{\rho\theta}{\ell}.$$

Consequently the shearing strains of the longitudinal fibers vary linearly as the distances from the center of the shaft.

If we assume that we are concerned only with the linear range of action of the material where the shearing stress is proportional to shearing strain, then it is evident that the shearing stresses of the longitudinal fibers vary linearly as the distances from the center of the shaft. Obviously the distribution of shearing stresses is symmetric around the geometric axis of the shaft. They have the appearance shown in Fig. 9.5. For equilibrium, the sum of the moments of these distributed shearing forces over the entire circular cross section is equal to the applied twisting moment. Also, the sum of the moments of these forces is exactly equal to the torque T shown in Fig. 9.7, b above.

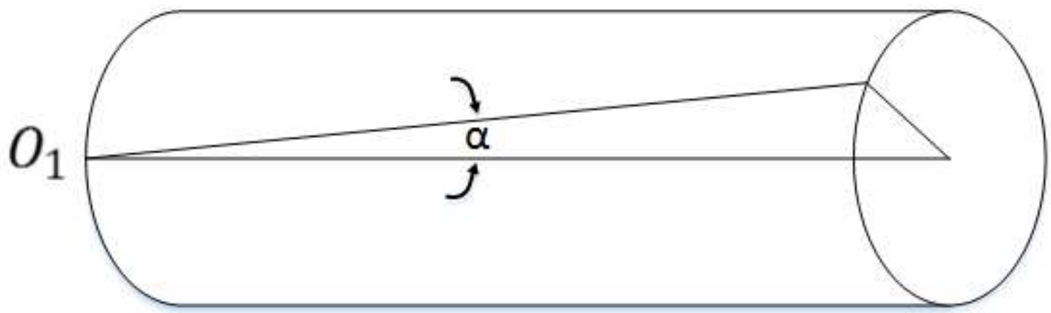

Fig. 9.5

Thus we have:

$$T = \int_0^r (\tau \cdot \rho) dA,$$

where dA represents the area of the shaded ring-shaped element shown in Fig. 9.5. However, the shearing stresses vary as the distances from the geometric axis; hence

$$\frac{\tau}{\rho} = const,$$

where the subscripts on the shearing stress denote the distances of the element from the axis of the shaft.

Consequently we may write:

$$T = \int_0^r \frac{\tau}{\rho} \left(\rho^2\right) dA = \frac{\tau}{\rho} \int_0^r \rho^2 dA,$$

since the ratio $\frac{\tau}{\rho}$ is a constant. However, the expression $\int_0^r \rho^2 dA$ is by definition the polar moment of inertia of the cross-sectional area. Hence the desired relationship is:

$$T = \frac{\tau \cdot \rho}{I_\rho},$$

or

$$\tau = \frac{T \cdot \rho}{I_\rho}.$$

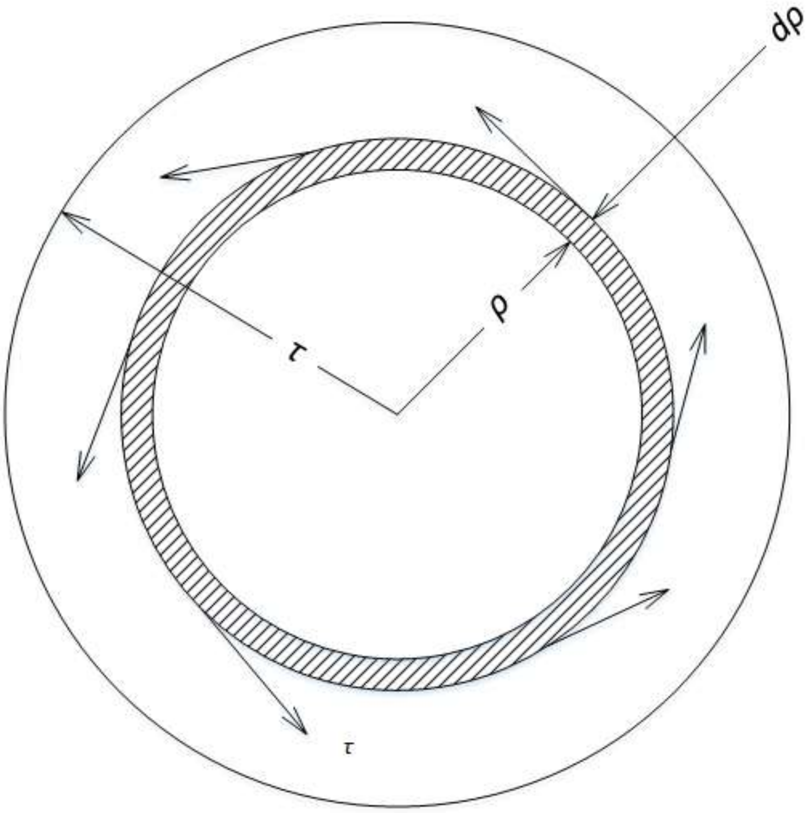

Fig. 9.6

It is to be emphasized that this expression holds only if no points of the bar are stressed beyond the proportional limit of the material.

If a generator $a - b$ is marked on the surface of the unloaded bar, then after the twisting moment T has been applied this line moves to $a - b$, as shown in Fig. 9.6. The angle γ, measured in radians, between the final and original positions of the generator is defined as the shearing strain at the surface of the bar. The same definition would hold at any interior point of the bar.

On beginning

9.2. About the angle of twist.

The ratio of the shear stress τ to the shear strain y is called the modulus of elasticity in shear is given by:

$$G = \frac{\tau}{\gamma}. \tag{9.3}$$

Again the units of G are the same as those of shear stress, since the shear strain is dimensionless.

Let us derive an expression for the angle of twist of a circular shaft as a function of the applied twisting moment. Assume that the entire shaft is acting within the elastic range of action of the material.

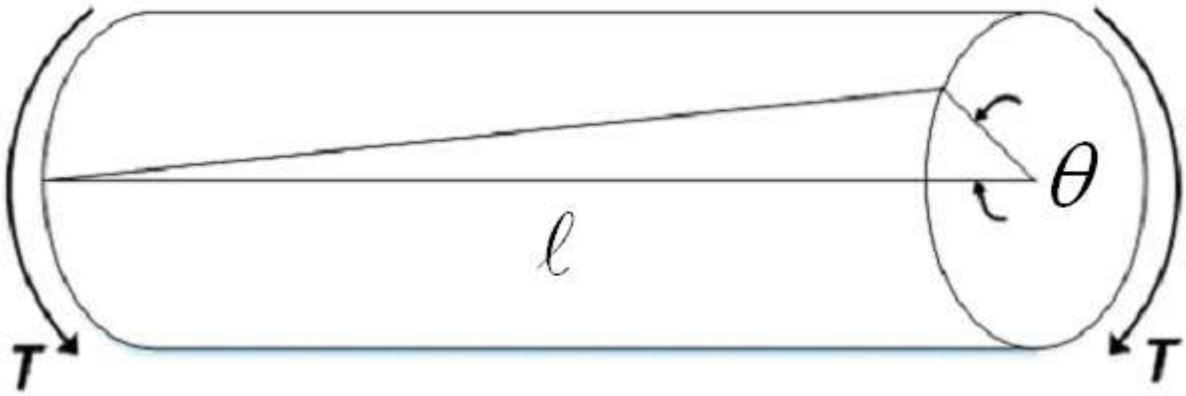

Fig. 9.7

Let ℓ denote the length of the shaft, I_ρ the polar moment of inertia of the cross section. T the applied twisting moment (assumed const ant along the length of the bar). and G the modulus of elasticity in shear. The angle of twist in a length ℓ is represented by θ in Fig. 9.7.

From Eq. (9.2) we have at the outer fibers where $\rho = r$:

$$\gamma = \frac{r\theta}{\ell} \qquad \text{and} \qquad \tau = \frac{T \cdot r}{I_\rho}.$$

By definition, the shear modulus is given by:

$$G = \frac{\tau}{\gamma} = \frac{T \cdot r}{I_\rho} \cdot \frac{\ell}{r\theta} = \frac{T \cdot \ell}{I_\rho \theta},$$

from which

$$\theta = \frac{T\ell}{GI_\rho}.$$

θ is expressed in radians, i.e., it is dimensionless.

Occasionally the angle of twist in a unit length is useful. It is often denoted by φ and is given by:

$$\varphi = \frac{\theta}{\ell} = \frac{T}{GI_\rho}. \qquad (9.4)$$

If a shaft of length ℓ is subject to a constant twisting moment T along its length, then the angle θ through which one end of the bar will twist relative to the other is:

$$\theta = \frac{T\ell}{GI_\rho}, \qquad (9.5)$$

where I_ρ denotes the polar moment of inertia of the cross section.

On beginning

9.3. The strength - weight ratio under torsion.

A shaft rotating with constant angular velocity ω (radians per second) is being acted on by a twisting moment T and hence transmits a power:

$$P = T \cdot \omega.$$

Alternatively, in terms of the number of revolutions per second f, the power transmitted is:

$$P = 2\pi f \cdot T.$$

As the twisting moment acting on either a solid or hollow circular bar is increased, a value of the twisting moment is finally reached for which the extreme fibers of the bar have reached the yield point in shear of the material. This is the maximum possible elastic twisting moment that the bar can withstand and is denoted by T_e.

A further increase in the value of the twisting moment puts the interior fibers at the yield point, with yielding progressing from the outer fibers inward. The limiting case occurs when all fibers are stressed to the yield point in shear and this represents the fully plastic twisting moment. It is denoted by T_ρ. Provided we do not consider stresses greater than the yield point in shear, this is the maximum possible twisting moment the bar can carry. For a solid circular bar subject to torsion:

$$T_\rho = \frac{4}{3} T_e.$$

Let us consider a thin-walled tube subject to torsion. Derive an approximate expression for the allowable twisting moment if the working stress in shear is a given constant τ. Also, derive an approximate expression for the strength-weight ratio or such a tube. It is assumed the tube does not buckle, and the material is within the elastic range or action.

The polar moment of inertia of a hollow circular shaft of outer diameter D and inner diameter d, is $I_\rho = \frac{\pi}{32}\left(D^4 - d^4\right)$. If R denotes the outer radius of the tube, then $D = 2R$, and further, if r denotes the wall thickness of the tube. then $d = 2R - 2t$.

The polar moment of inertia I_ρ may be written in the alternate form:

$$I_\rho = \frac{\pi}{32}\left((2R)^4 - (2R-2t)^4\right) = \frac{\pi}{2}\left(R^4 - (R-t)^4\right) =$$
$$= \frac{\pi}{2}\left(4R^3 t - 6R^2 t^2 + 4Rt^3 + t^4\right) =$$
$$= \frac{\pi R^4}{2}\left(4\left(\frac{t}{R}\right) - 6\left(\frac{t}{R}\right)^2 + 4\left(\frac{t}{R}\right)^3 + \left(\frac{t}{R}\right)^4\right).$$

Neglecting squares and higher powers of the ration $\frac{t}{R}$, since we are considering a thin-walled tube, this becomes, approximately, $I_\rho = 2\pi R^3 t$.

The ordinary torsion formula is:

$$T = \frac{\tau \cdot I_\rho}{R}.$$

For a thin - walled tube this becomes, for the allowable twisting moment:

$$T = 2\pi R^2 t\tau .$$

The weight W of the tube is:

$$W = \gamma \cdot \ell \cdot A,$$

where γ is the specific weight of the material, ℓ the length of the tube, and A the cross - sectional area of the tube. The area is given by:

$$A = \pi\left[R^2 - (R-t)^2\right] = \pi\left(2Rt - t^2\right) = \pi R^2\left(2\frac{t}{R} - \frac{t^2}{R^2}\right).$$

Again neglecting the square of the ratio $\frac{t}{R}$ for a thin tube, this becomes:

$$A = 2\pi Rt .$$

The strength - weight ratio is defined to be $\frac{T}{W}$. This is given by:

$$\frac{T}{W} = \frac{2\pi R^2 t\tau}{2\pi Rt\ell\gamma} = \frac{R\tau}{\ell\gamma}.$$

The ratio is of considerable importance in aircraft design.

On beginning